Compact schemes for linear partial differential equations

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A general approach to compact difference schemes' construction is developed. A differential problem

$$Au = f \text{ or } Au = Bf$$

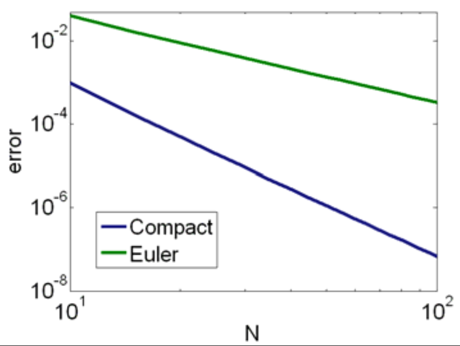
will approximated by difference one

$$Pu_h = Qf_h. \quad (1)$$

We want determine the "optimal" difference operators P and Q. In the simplest problem $Au = d_x^2 u = f$ we can search the three-point difference operators P and Q and obtain the 4-th order compact difference scheme (CDS): $au_{j-1} + bu_j + cu_{j+1} = pf_{j-1} + qf_j + rf_{j+1}$, with coefficients $a = c = 1;\ b = -2;\ p = r = h^2/12;\ q = 5h^2/6$.

To determine the coefficients of the scheme we can assume that the following test functions: $u_k = x^k$, $f_k = Au_k$, k=0..4 are exact solutions of (1). Thus we obtain the coefficients.

The standard difference scheme $a = c = 1;\ b = -2;\ p = r = 0;\ q = h^2$. The order of the standard scheme is equal to 2, only.



Any case we can use the double-sweep method to solve both linear algebraic systems with three-diagonal matrix. But the error of the compact scheme is much smaller.

Here N is the number of the grid points with respect to x.

The following typical for mathematical physics operators A are considered:

- Laplace op. (Δ);
- Helmholtz op. ($\Delta - q(\vec{x})$);
- diffusion op. with constant diffusion coefficient ($\partial_t - D\Delta - q(\vec{x})$);
- diffusion op. with variable one ($\partial_t - \partial_x D \partial_x - q(\vec{x})$);
- Schrodinger op. ($\partial_t - iD\Delta - q(\vec{x})$);
- rod vibrations op. ($\partial_t^2 - D\partial_t^2\partial_x^2 + C\partial_x^4$).

We constructed 4-th order CDSs and confirmed this order by numerical experiments for various boundary, initial-boundary, and eigen-values problems. We compare these CDSs with classic ones, e. g. with the Crank – Nicolson scheme. The relative high-order approximations for corresponding boundary and initial conditions were constructed, too.

1. *V.A.Gordin. Mathematics, Computer, Weather Forecasting, and Other Scenarios of Mathematical Physics (in Russian). M., Fizmatlit, 2010, 2012.*
2. *V. A.Gordin, E.A. Tsymbalov. Compact Difference Schemes for the Diffusion and Schrödinger Equations. Approximation, Stability, Convergence, Effectiveness, Monotony. Journal of Computational Mathematics, 2014, Vol. 32. N 3, pp.348-370.*